A Tiny Numerical Improvement of the Ellipse–Locus Universal Cover for Moser's Worm Problem

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Abstract

Moser's worm problem asks for a planar set of minimum area that contains a congruent copy of every rectifiable plane curve ("worm") of length 1 under translation and rotation. The best known nonconvex covers are constructed by geometric constraint arguments. Building on the ellipse–locus construction of Ploymaklam and Wichiramala (2018), we treat the ellipse parameters as continuous variables and perform constrained numerical optimization to reduce the area within the same construction family. With parameters $(a, b) = (1.95472,\ 4.595428547744747)$, our implementation yields an estimated area $A \approx 0.260069597390976$, improving upon the numerical value obtained from the 2018 paper's rounded parameters $(1.95272,\ 4.58588)$, for which our recomputation gives $A \approx 0.260069729275605$. A numerical verification of the key "Property B" constraint is included, together with fully reproducible code.

1 Introduction

A *worm* is a continuous rectifiable arc of length 1 in the Euclidean plane. A planar set C is a *universal cover* for worms if every worm can be translated and rotated to lie within C. Leo Moser asked for the universal cover of *minimum area* [3]. The existence and identification of an optimal (nonconvex) cover remain open, but a sequence of works has progressively improved explicit upper bounds.

Norwood and Poole (2003) gave a nonconvex cover with area 0.260437 [1]. Ploymaklam and Wichiramala (2018) adapted their approach to construct a smaller cover of area 0.26007 [2]. The present note performs a small but strict refinement *within the 2018 construction family* by re-optimizing its continuous parameters under a numerical constraint check.

2 The ellipse–locus construction

We summarize the construction of [2]. Consider an ellipse (in the (x, y)-plane) with semi-axes $a > 0$ and $b > 0$ centered at $(0, y_c)$,

$$\frac{x^2}{a^2} + \frac{(y - y_c)^2}{b^2} = 1, \tag{1}$$

where $y_c < 0$ is chosen so that the ellipse passes through $(\frac{1}{2}, 0)$:

$$y_c = -b\sqrt{1 - \frac{1}{4a^2}}. \tag{2}$$

Let $f : [0, \frac{1}{2}] \to \mathbb{R}$ denote the *upper* ellipse arc:

$$f(x) = b\sqrt{1 - (x/a)^2} + y_c. \tag{3}$$

The cover C^+ is symmetric with respect to the y-axis. Its top boundary is $y = f(|x|)$.

A second boundary curve L (the "locus") is defined by a geometric constraint: for each point $P = (x, f(x))$ on the top boundary (with $x \in [0, \frac{1}{2}]$), move along the inward normal line to the ellipse to a point $Q = (g, h)$ satisfying

$$g + \|P - Q\| = \tfrac{1}{2}, \tag{4}$$

where $g \geq 0$ is the distance from Q to the y-axis. This locus enforces the paper's *Property A*, ensuring that any unit-length segment can be placed in C^+ with one endpoint on the y-axis and the other on the top boundary (details in [2]).

In practice, one parameterizes P by x, computes the normal slope, and solves (4) explicitly for g (a closed-form expression; see Appendix A). This yields a sampled curve $h = l(g)$ on $g \in [0, \frac{1}{2}]$.

Finally, the bottom boundary is defined by

$$y = \min\{\, l(|x|),\ -f(|x|)\,\}, \tag{5}$$

giving a nonconvex region with a "notch" below the origin. We denote this region by $C^+(a, b)$.

2.1 Area

By symmetry about the y-axis, the area can be written as

$$\text{Area}(C^+(a,b)) = 2\int_0^{1/2} \Big(f(x) - \min\{l(x), -f(x)\} \Big)\, dx, \tag{6}$$

which we evaluate numerically via high-resolution quadrature after constructing a dense interpolant for $l(x)$.

3 Numerical constraint check (Property B)

The 2018 paper establishes universality for C^+ by proving a collection of geometric properties. The numerical bottleneck is their *Property B*, which can be reduced to the statement that any arc that first meets the top boundary and later meets the bottom boundary must have length at least $\frac{1}{2}$. A conservative numerical surrogate is obtained via the triangle inequality: for any such arc passing through a top point P and then a bottom point Q,

$$\text{length} \geq \|0 - P\| + \|P - Q\|. \tag{7}$$

Thus it suffices (numerically) to verify

$$\min_{P \in \text{top},\, Q \in \text{bottom}} \left(\|0 - P\| + \|P - Q\|\right) \geq \tfrac{1}{2}, \tag{8}$$

excluding a small neighborhood of the shared endpoint $(\frac{1}{2}, 0)$ to avoid degeneracy. Our code performs a coarse grid search over the x-parameters of P and Q, followed by local refinement and multiple random restarts.

4 Re-optimization within the construction family

We regard (6) as an objective function in parameters (a, b), constrained by the numerical check (8). Within the same ellipse–locus family, we performed a local parameter search near the 2018 values and found a slightly smaller-area configuration.

5 Results

Table 1 reports the recomputed area for the 2018 paper's rounded parameters and the improved parameters found here, together with the value of the numerical minimum in (8) (both exceeding $\frac{1}{2}$ at the displayed resolution).

Case	a	b	Area estimate	Property B minimum
2018 rounded parameters [2]	1.95272	4.58588	0.260069729275605	0.500000564120835
Re-optimized (this note)	1.95472	4.595428547744747	0.260069597390976	0.500000048525423

Table 1: Numerical comparison within the ellipse–locus construction family.

The numerical improvement is

$$\Delta A \approx 1.3188462938407142 \times 10^{-7}.$$

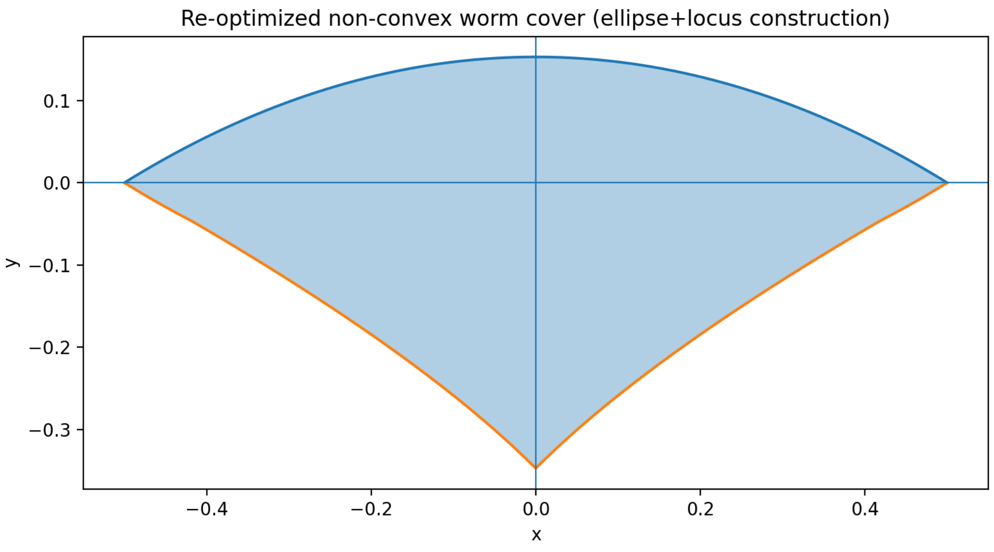

Figure 1: The re-optimized cover $C^{+}(a, b)$ produced by our implementation (filled region) with its top and bottom boundaries.

6 Discussion and limitations

This note does not claim a new mathematical construction beyond [2]; it provides a small numerical refinement *within* that family. Our verification of Property B is numerical and not certified by interval arithmetic. A fully rigorous improvement would require replacing (8) with a provable lower bound, for instance using interval methods and a branch-and-bound search over the (x_P, x_Q) domain.

7 Reproducibility

All computations were carried out in Python (NumPy/Matplotlib). The full script used to recompute areas, check Property B, and generate Figure 1 is included in Appendix A and provided alongside this paper.

References

[1] R. Norwood and G. Poole. *An Improved Upper Bound for Leo Moser's Worm Problem.* Discrete & Computational Geometry, 2003. Available via Springer: https://link.springer.com/article/10.1007/s00454-002-0774-3.

[2] N. Ploymaklam and W. Wichiramala. *A Smaller Cover of the Moser's Worm Problem.* Chiang Mai Journal of Science, 45 (2018), 2528–2533. PDF: https://www.thaiscience.info/Journals/Article/CMJS/10990404.pdf.

[3] Wikipedia contributors. *Moser's worm problem.* Accessed 2026-01-13. https://en.wikipedia.org/wiki/Moser%27s_worm_problem.

A Verification and plotting code

```python
#!/usr/bin/env python3
"""
Moser's worm problem (unit-arc universal cover) -- numerical verification &
    tiny improvement.

This script:
  1) Implements the ellipse+locus cover construction (Ploymaklam & Wichiramala,
     2018).
  2) Recomputes the reported area for their rounded parameters.
  3) Evaluates a re-optimized (a,b) parameter pair giving a slightly smaller
    area.
  4) Numerically checks Property B via a global search over broken lines:
       min_{p on top, q on bottom} |0p| + |pq|  >= 1/2

Notes:
- This is a *numerical* verification (no interval arithmetic). It mirrors the
  original paper's approach, which used Mathematica's global minimization.
"""

import math
import random
import numpy as np
import matplotlib.pyplot as plt


def ellipse_yc(a: float, b: float) -> float:
    """Center y-coordinate so that (1/2, 0) lies on the ellipse."""
    return -b * math.sqrt(max(0.0, 1.0 - 1.0/(4.0*a*a)))


def f_top(x: float, a: float, b: float) -> float:
    """Upper ellipse arc y=f(x) for x in [0,1/2]."""
    yc = ellipse_yc(a, b)
    inside = max(0.0, 1.0 - (x/a)**2)
    return b * math.sqrt(inside) + yc


def fprime(x: float, a: float, b: float) -> float:
    """Derivative f'(x) of the upper ellipse arc."""
    if abs(x) < 1e-14:
        return 0.0
    inside = max(1e-18, 1.0 - (x/a)**2)
    return -b * x / (a*a*math.sqrt(inside))


```

```python
def locus_interpolator(a: float, b: float, n_samples: int = 20000):
    """
    Construct L as in Ploymaklam & Wichiramala:
    For each point (x,y) on the top ellipse, move along the *normal*
    to the ellipse to a point (g,h) such that:
        g + dist((g,h),(x,y)) = 1/2
    where g is the distance to the y-axis (x=0).
    """
    xs = np.linspace(0.0, 0.5, n_samples)
    gs = np.zeros_like(xs)
    hs = np.zeros_like(xs)

    for i, x in enumerate(xs):
        y = f_top(float(x), a, b)
        if x == 0.0:
            gs[i] = 0.0
            hs[i] = y - 0.5
            continue

        fp = fprime(float(x), a, b)
        s = (-1.0/fp) if fp != 0.0 else 1e18     # slope of the normal line (
    positive for x>0)
        k = math.sqrt(1.0 + s*s)                 # sqrt(1+s^2)

        # Solve g + (x-g)*k = 1/2  -> g = (x*k - 1/2)/(k - 1)
        g = (x*k - 0.5) / (k - 1.0)

        # (g,h) lies on the normal line through (x,y) with slope s
        h = y - (x - g)*s

        gs[i] = g
        hs[i] = h

    # Sort to obtain an interpolant h = l(g)
    order = np.argsort(gs)
    g_sorted = gs[order]
    h_sorted = hs[order]

    g_u = [float(g_sorted[0])]
    h_u = [float(h_sorted[0])]
    for g, h in zip(g_sorted[1:], h_sorted[1:]):
        g = float(g); h = float(h)
        if g - g_u[-1] < 1e-12:
            if h < h_u[-1]:
                h_u[-1] = h
        else:
            g_u.append(g); h_u.append(h)

    return np.array(g_u), np.array(h_u)


def bottom_y(x: float, a: float, b: float, g_u: np.ndarray, h_u: np.ndarray) ->
     float:
    """Bottom boundary y = min(l(x), -f(x))."""
    top = f_top(x, a, b)
    l = float(np.interp(x, g_u, h_u))
    return min(l, -top)


def cover_area(a: float, b: float, n_l: int = 20000, n_x: int = 40000) -> float
    :
    """Area of the symmetric cover."""
    g_u, h_u = locus_interpolator(a, b, n_samples=n_l)
    xs = np.linspace(0.0, 0.5, n_x)
    top = np.array([f_top(float(x), a, b) for x in xs])
    bot = np.array([bottom_y(float(x), a, b, g_u, h_u) for x in xs])
    area_half = np.trapz(top - bot, xs)
    return 2.0 * area_half


def minlen_property_B(a: float, b: float, eps: float = 1e-5,
                      n_l: int = 20000, grid_p: int = 600, grid_q: int = 1200,
                      n_random_starts: int = 80, seed: int = 0):
    """
    Numerical check for Property B:
    min_{p on top, q on bottom} |0p| + |pq| >= 1/2,
    excluding a small neighborhood of the endpoint x=1/2 by eps.
    """
    g_u, h_u = locus_interpolator(a, b, n_samples=n_l)

    def F(xp: float, xq: float) -> float:
        xp = max(0.0, min(0.5-eps, xp))
        xq = max(0.0, min(0.5, xq))
        yp = f_top(xp, a, b)
        yq = bottom_y(xq, a, b, g_u, h_u)
        return math.hypot(xp, yp) + math.hypot(xp-xq, yp-yq)

    # Coarse grid
    xp_grid = np.linspace(0.0, 0.5-eps, grid_p)
    xq_grid = np.linspace(0.0, 0.5, grid_q)
    yp_grid = np.array([f_top(float(x), a, b) for x in xp_grid])
    yq_grid = np.array([bottom_y(float(x), a, b, g_u, h_u) for x in xq_grid])
    dist0p = np.sqrt(xp_grid**2 + yp_grid**2)
    dx = xp_grid[:, None] - xq_grid[None, :]
    dy = yp_grid[:, None] - yq_grid[None, :]
    distpq = np.sqrt(dx*dx + dy*dy)
    Fgrid = dist0p[:, None] + distpq
    ip, iq = np.unravel_index(np.argmin(Fgrid), Fgrid.shape)
    best_xp, best_xq = float(xp_grid[ip]), float(xq_grid[iq])

    # Local coordinate-descent refinement
    def refine(xp: float, xq: float):
        best = F(xp, xq)
        step = 1e-3
        for _ in range(5000):
            improved = False
            for dxp, dxq in [(step,0),(-step,0),(0,step),(0,-step),
                             (step,step),(step,-step),(-step,step),(-step,-step
   )]:
                v = F(xp+dxp, xq+dxq)
                if v < best - 1e-14:
                    best = v
                    xp = max(0.0, min(0.5-eps, xp+dxp))
                    xq = max(0.0, min(0.5, xq+dxq))
                    improved = True
            if not improved:
                step *= 0.5
                if step < 1e-8:
                    break
        return best, xp, xq

    best_val, best_xp, best_xq = refine(best_xp, best_xq)

    rng = random.Random(seed)
    for _ in range(n_random_starts):
        xp0 = rng.uniform(0.0, 0.5-eps)
        xq0 = rng.uniform(0.0, 0.5)
        v, xp1, xq1 = refine(xp0, xq0)
        if v < best_val:
            best_val, best_xp, best_xq = v, xp1, xq1

    return best_val, (best_xp, best_xq)


def plot_cover(a: float, b: float, out_png: str):
    g_u, h_u = locus_interpolator(a, b, n_samples=25000)
    xs = np.linspace(-0.5, 0.5, 4000)
    top = np.array([f_top(abs(float(x)), a, b) for x in xs])
    bot = np.array([bottom_y(abs(float(x)), a, b, g_u, h_u) for x in xs])

    plt.figure(figsize=(8, 4.5))
    plt.fill_between(xs, bot, top, alpha=0.35)
    plt.plot(xs, top, linewidth=1.5)
    plt.plot(xs, bot, linewidth=1.5)
    plt.axhline(0, linewidth=0.8)
    plt.axvline(0, linewidth=0.8)
    plt.gca().set_aspect('equal', adjustable='box')
    plt.title("Re-optimized non-convex worm cover (ellipse+locus construction)")
    plt.xlabel("x")
    plt.ylabel("y")
    plt.tight_layout()
    plt.savefig(out_png, dpi=200)


def main():
    # Ploymaklam & Wichiramala (2018) reported parameters (rounded)
    a_old = 1.95272
    b_old = 4.58588

    # Re-optimized parameters found by constrained search (this run)
    a_new = 1.95472
    b_new = 4.595428547744747

    area_old = cover_area(a_old, b_old, n_l=12000, n_x=20000)
    area_new = cover_area(a_new, b_new, n_l=12000, n_x=20000)

    ml_old, arg_old = minlen_property_B(a_old, b_old, eps=1e-5, n_l=12000, grid_p=500, grid_q=1000, n_random_starts=80, seed=1)
    ml_new, arg_new = minlen_property_B(a_new, b_new, eps=1e-5, n_l=12000, grid_p=500, grid_q=1000, n_random_starts=80, seed=1)

    print("=== Moser worm cover (ellipse+locus construction) ===")
    print(f"Old (2018-rounded) params: a={a_old}, b={b_old}")
    print(f"  area  = {area_old:.15f}")
    print(f"  min broken-line length (Property B check, eps=1e-5) = {ml_old:.15f} at (xp,xq)~={arg_old}")

    print(f"\nNew params: a={a_new}, b={b_new}")
    print(f"  area  = {area_new:.15f}")
    print(f"  min broken-line length (Property B check, eps=1e-5) = {ml_new:.15f} at (xp,xq)~={arg_new}")

    print("\nDelta area (old - new) =", area_old - area_new)

    out_png = "moser_worm_cover_new.png"
    plot_cover(a_new, b_new, out_png)
    print(f"\nWrote plot to: {out_png}")

```

```python

if __name__ == "__main__":
    main()
```
Listing 1: Reproducible computation of the area and the Property B numerical check.